A Level Set Approach to Robust Integer Nonlinear Optimization

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Outline

Planar Barycenter Problem

Given $A_1, \ldots, A_M$ and weights $\omega_1, \ldots, \omega_M > 0$:

$$(Bary) \quad \min \quad \sum_{m=1}^{M} \omega_m \|A_m - x\|_2^2$$
$$\text{s.t.} \quad x \in \mathbb{R}^2.$$

Planar Barycenter Problem

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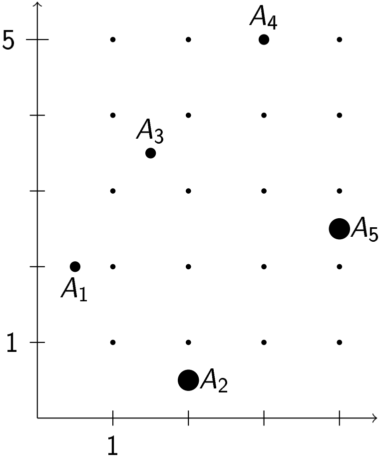
$$(Bary) \quad \min \sum_{m=1}^{M} \omega_m \|A_m - x\|_2^2 \quad \text{s.t.} \quad x \in \mathbb{R}^2.$$

Example: $\omega = (1, 2, 1, 1, 2)$.



Planar Barycenter Problem: Eyster, White '73

$$(Bary) \quad \min \quad \sum_{m=1}^{M} \omega_m \|A_m - x\|_2^2$$
$$\text{s.t.} \quad x \in \mathbb{R}^2$$

is solved to optimality by

$$x^* = \frac{1}{\sum_{m=1}^{M} \omega_m} \left(\sum_{m=1}^{M} \omega_m A_{m,1}, \sum_{m=1}^{M} \omega_m A_{m,2} \right)$$

and the level sets of the objective function are concentric circles centered on x^*.

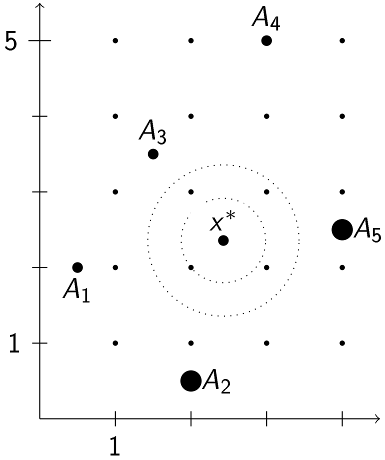
Level set

Definition

Given a function $f: \mathbb{R}^n \to \mathbb{R}$ we denote the level set with respect to some level $z \in \mathbb{R}$ by

$$\mathcal{L}_{\leq}(z) := \{x \in \mathbb{R}^n : f(x) \leq z\}.$$

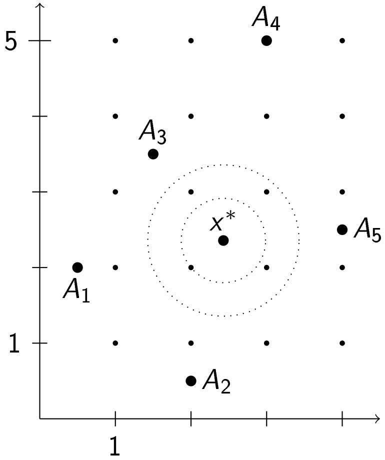
(Bary) - Example



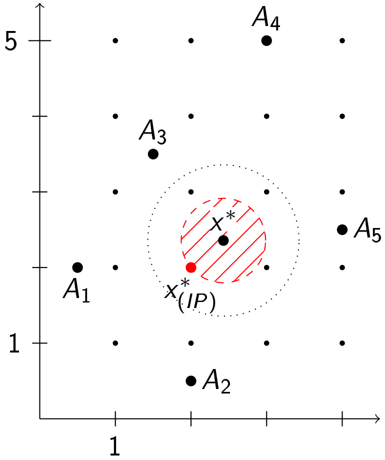
Integer (Bary)

$$(IBary) \quad \min \quad \sum_{m=1}^{M} \omega_m \|A_m - x\|_2^2$$
$$\text{s.t.} \quad x \in \mathbb{Z}^2.$$

Integer (Bary) - Example



Integer (Bary) - Example



Outline

In general

We consider

$$(IP) \quad \min \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{Z}^n$$

with continuous relaxation

$$(CP) \quad \min \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n.$$

$x^*_{(IP)}$: a minimizer to (IP)
x^*: a minimizer to (CP).

Rounding property

Definition (Hübner, Schöbel '11)

A function $f: \mathbb{R}^n \to \mathbb{R}$ has the *rounding property* if: For any optimal solution x^* to (CP) there exists an optimal solution $x^*_{(IP)}$ to (IP), such that

$$x^*_{(IP)} \in \text{Round}(x^*).$$

Where

$$\text{Round}(x) := \{y \in \mathbb{Z}^n : \; y_i \in \{\lfloor x_i \rfloor, \lceil x_i \rceil\} \;\; \forall i\}.$$

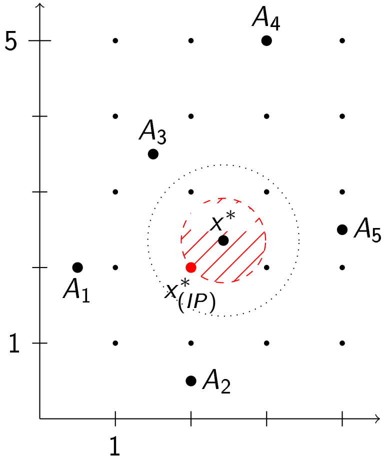
Level set approach

Lemma (Hübner, Schöbel '11)

f has the rounding property $\Leftrightarrow$ for any optimal solution x^* to (CP) and for all $x \in \mathbb{Z}^n$ we have that

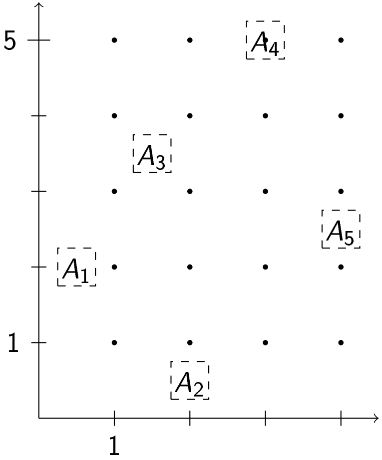
$$\mathcal{L}_{\leq}(f(x)) \cap \text{Round}(x^*) \neq \emptyset.$$

Barycenter Problem - Example



Outline

Uncertain (Bary)



Robust Optimization: Ben-Tal, Nemirovski '98

Uncertain optimization problem:

$$
\begin{array}{lll}
(P(\xi)) & \min & f(x,\xi) \\
 & \text{s.t.} & x \in \mathbb{R}^n,
\end{array}
$$

with some uncertain parameter $\xi \in U$.

Robust Optimization: Ben-Tal, Nemirovski '98

Uncertain optimization problem:

$$(P(\xi)) \quad \min \quad f(x, \xi) \quad \text{s.t.} \quad x \in \mathbb{R}^n,$$

with some uncertain parameter $\xi \in U$.

Literature on Robust Location Theory: see for example Snyder '06.

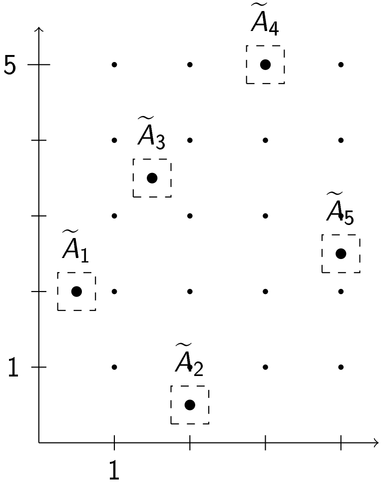
(Bary) - Uncertainty set 1

In our example:

$$\xi = (\widetilde{A}_1, \ldots, \widetilde{A}_5) \in \bigtimes_{i=1}^{5} \left([A_{i1} - \epsilon, A_{i1} + \epsilon] \times [A_{i2} - \epsilon, A_{i2} + \epsilon] \right)$$

(where $A_1, \ldots, A_5$ are the nominal points).

(Bary) - Uncertainty set 1



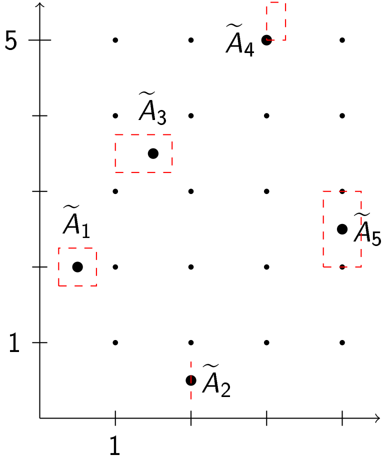
(Bary) - Other Uncertainty sets

More general:

$$\xi = (\widetilde{A}_1, \ldots, \widetilde{A}_5) \in \bigtimes_{i=1}^{5} \left([A_{i1} - \epsilon_1^i, A_{i1} + \epsilon_2^i] \times [A_{i2} - \epsilon_3^i, A_{i2} + \epsilon_4^i]\right)$$

(where $A_1, \ldots, A_5$ are the nominal points).

(Bary) - Uncertainty set 2



(Bary) - Uncertainty set 3

Other example:

$$\xi = (\widetilde{\omega}_1, \ldots, \widetilde{\omega}_5) \in \bigtimes_{i=1}^{5} [\omega_i - \epsilon_1^i, \omega_i + \epsilon_2^i]$$

Robust Counterpart

Definition (Ben-Tal, Nemirovski '98)

The robust counterpart to a problem of form $(P(\xi))$ is given by

$$
\begin{aligned}
(RC) \quad & \min \quad \sup_{\xi \in U} f(x, \xi) \\
& \text{s.t.} \quad x \in \mathbb{R}^n.
\end{aligned}
$$

Basic Idea

Original Problem:

$$(IP(\xi)) \quad \min \quad f(x,\xi)$$
$$\text{s.t.} \quad x \in \mathbb{Z}^n$$

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$$\begin{array}{lll} (IP(\xi)) & \min & f(x,\xi) \\ & \text{s.t.} & x \in \mathbb{Z}^n \end{array}$$

$$\downarrow$$

Robust Counterpart:

$$\begin{array}{lll} (IRC) & \min & \hat{f}(x) \\ & \text{s.t.} & x \in \mathbb{Z}^n \end{array}$$

where $\hat{f}(x) := \sup_{\xi \in U} f(x,\xi)$.

Basic Idea

Original Problem:

$$(IP(\xi)) \quad \min \quad f(x,\xi) \quad \text{s.t.} \quad x \in \mathbb{Z}^n$$

$$\downarrow$$

Robust Counterpart:

$$(IRC) \quad \min \quad \hat{f}(x) \quad \text{s.t.} \quad x \in \mathbb{Z}^n$$

$$\longrightarrow$$

Relaxation:

$$(CRC) \quad \min \quad \hat{f}(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n$$

where $\hat{f}(x) := \sup_{\xi \in U} f(x,\xi)$.

Basic Idea

Original Problem:

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Robust Counterpart:

$$(IRC) \quad \min \quad \hat{f}(x) \quad \text{s.t.} \quad x \in \mathbb{Z}^n$$

$$\longrightarrow$$

Relaxation:

$$(CRC) \quad \min \quad \hat{f}(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n$$

$$\longleftarrow$$

Rounding Property

where $\hat{f}(x) := \sup_{\xi \in U} f(x,\xi)$.

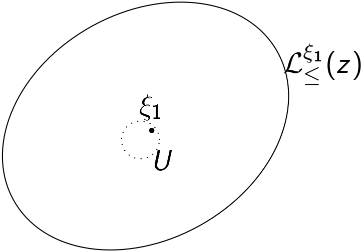
Outline

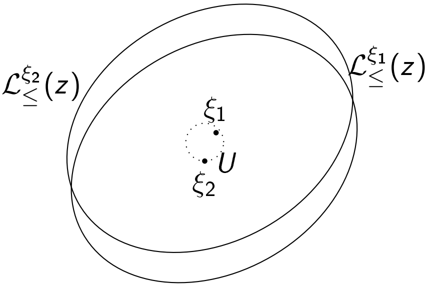
Level sets of $\hat{f} := \sup_{\xi \in U} f(x, \xi)$

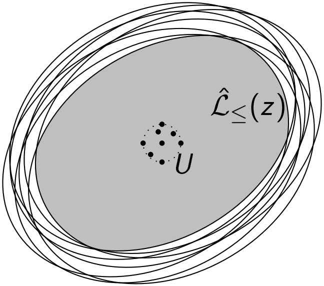
Lemma

Given $z \in \mathbb{R}$ and $\mathcal{L}_{\leq}^{\xi}(z) = \{x \in \mathbb{R}^n : \ f(x, \xi) \leq z\}$ for all $\xi \in U$, the level set $\hat{\mathcal{L}}_{\leq}(z)$ of $\hat{f}$ is of the form

$$\hat{\mathcal{L}}_{\leq}(z) = \bigcap_{\xi \in U} \mathcal{L}_{\leq}^{\xi}(z).$$


$\mathcal{L}^{\xi_1}_{\leq}(z)$
ξ_1
U


$\mathcal{L}^{\xi_2}_{\leq}(z)$
$\mathcal{L}^{\xi_1}_{\leq}(z)$
ξ_1
U
ξ_2


$\hat{\mathcal{L}}_{\leq}(z)$
U

Reduction to *Finite* Uncertainty

Theorem

If U is a bounded polytope and $f(x,\xi)$ is quasiconvex in x it holds that

$$\hat{\mathcal{L}}_{\leq}(z) := \bigcap_{\xi \in U} \mathcal{L}^{\xi}_{\leq}(z) = \bigcap_{\xi \in ext(U)} \mathcal{L}^{\xi}_{\leq}(z).$$

Equal Round Level Sets, Box-shaped U

Lemma

Let $f(x,\xi) = c \cdot \|x-\xi\|_2$ for all $\xi \in U = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Then $\hat{\mathcal{L}}_{\leq}(z)$ is cross-shaped w.r.t. continuous robust optimum x^*_{CRC} for all $z \in \mathbb{R}$.

Cross-shaped

Definition (Hübner, Schöbel '11)

We call a set $M \subseteq \mathbb{R}^n$ cross-shaped w.r.t. a center x_0 if for every $y \in M$ the box

$$[x_0, y]_{l_1} \subseteq M.$$

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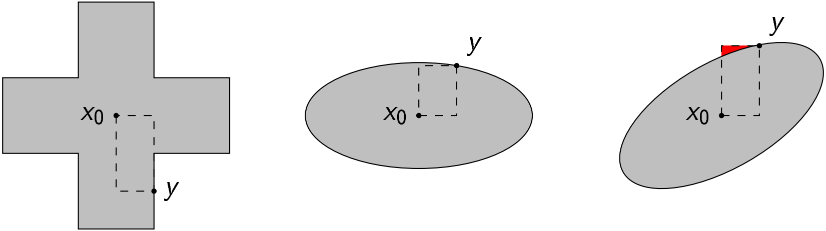
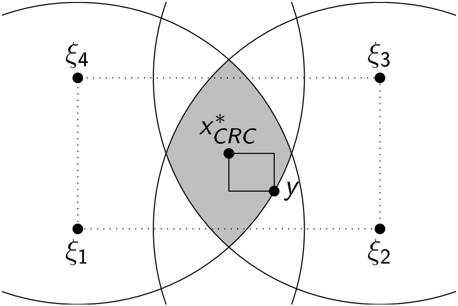


cross-shaped cross-shaped not cross-shaped

Equal Round Level Sets, Box-shaped U



Cross-shaped

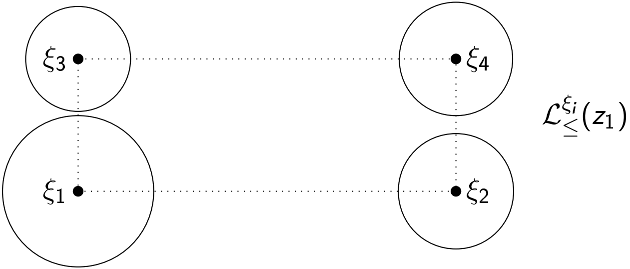
Theorem (Hübner, Schöbel '11)

Cross-shapedness of the level sets $\mathcal{L}_{\leq}(z)$ w.r.t. any continuous minimum x^*_{CRC} for all $z \leq \min\{\hat{f}(x) : x \in \text{Round}(x^*_{CRC})\}$ implies the rounding property.

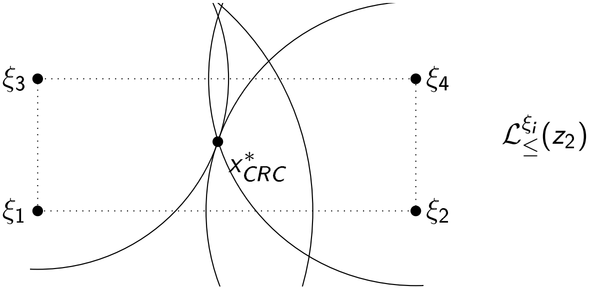
Corollary

Let $f(x, \xi) = c \cdot \|x - \xi\|_2$ for all $\xi \in U = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Then $\hat{f}$ has the rounding property.

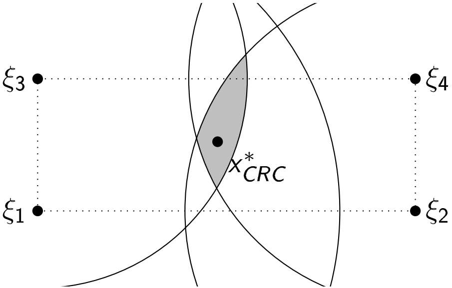
Variable Round Level Sets, Box-shaped U



Variable Round Level Sets, Box-shaped U

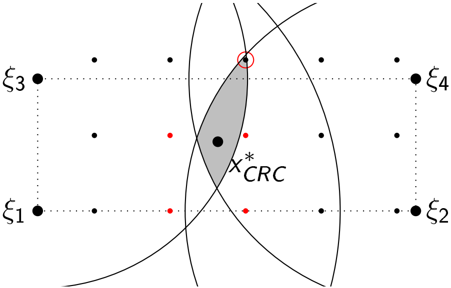


Variable Round Level Sets, Box-shaped U



$\mathcal{L}^{\xi_i}_{\leq}(z_3)$

Variable Round Level Sets, Box-shaped U



$\mathcal{L}^{\xi_i}_{\leq}(z_4)$

Almost Equal Round Level sets, Small U

Let

$$\mathcal{L}^{\xi}_{\leq}(z) = B_{\xi}(r^{\xi}(z)) := \{x \in \mathbb{R}^n : \ \|x - \xi\|_2 \leq r^{\xi}(z)\}$$

with $\underline{r}(z) \leq r^{\xi}(z) \leq \overline{r}(z)$ for all $\xi \in U$.

Almost Equal Round Level sets, Small U

Let

$$\mathcal{L}_{\leq}^{\xi}(z) = B_{\xi}(r^{\xi}(z)) := \{x \in \mathbb{R}^n : \ \|x - \xi\|_2 \leq r^{\xi}(z)\}$$

with $\underline{r}(z) \leq r^{\xi}(z) \leq \overline{r}(z)$ for all $\xi \in U$.
Then

$$\bigcap_{\xi \in U} B_{\xi}(\underline{r}(z)) \subseteq \hat{\mathcal{L}}_{\leq}(z) \subseteq \bigcap_{\xi \in U} B_{\xi}(\overline{r}(z)).$$

Almost Equal Round Level sets, Small U

Let

$$\mathcal{L}^{\xi}_{\leq}(z) = B_{\xi}(r^{\xi}(z)) := \{x \in \mathbb{R}^n : \ \|x - \xi\|_2 \leq r^{\xi}(z)\}$$

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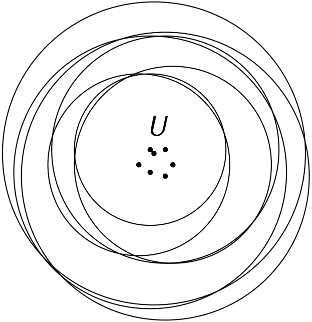
Then

$$\bigcap_{\xi \in U} B_{\xi}(\underline{r}(z)) \subseteq \hat{\mathcal{L}}_{\leq}(z) \subseteq \bigcap_{\xi \in U} B_{\xi}(\overline{r}(z)).$$

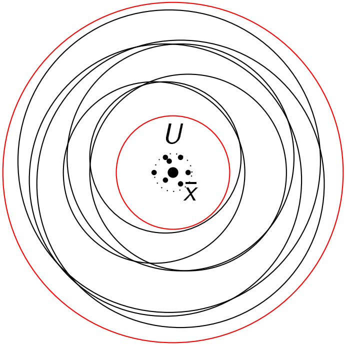
Let $diam(U) \leq 2\epsilon \Rightarrow \exists \ \bar{x}$ s.t. $U \subseteq B_{\bar{x}}(\epsilon)$.

$$\Rightarrow \quad B_{\bar{x}}(\underline{r}(z) - \epsilon) \subseteq \hat{\mathcal{L}}_{\leq}(z) \subseteq B_{\bar{x}}(\overline{r}(z) + \epsilon).$$

Example



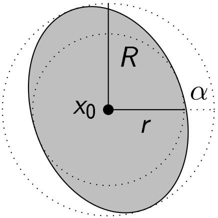
Example



Quasiround

Definition (Hübner, Schöbel ’11)

Given $\alpha \geq 0$ we call a set $M \subseteq \mathbb{R}^n$ α-quasiround w.r.t. a center x_0 if there exist $r \in \mathbb{R}_0^+$ and $R \in \mathbb{R}^+$ s.t. $B_{x_0}(r) \subseteq M \subseteq B_{x_0}(R)$ and $R - r \leq \alpha$.



Quasiround

Theorem (Hübner, Schöbel '11)

α-Quasiroundness of the level sets for $\alpha = \frac{1}{2}\left(\sqrt{n+3} - \sqrt{n-1}\right)$ w.r.t. the continuous minimum for all $z \le \min\{f(x) : x \in \text{Round}(x^*)\}$ implies the rounding property.

Almost Equal Round Level sets, Small U

Corollary

Let

$$\mathcal{L}^{\xi}_{\leq}(z) = B_{\xi}(r^{\xi}(z)) := \{x \in \mathbb{R}^n : \ \|x - \xi\|_2 \leq r^{\xi}(z)\}$$

with $\underline{r}(z) \leq r^{\xi}(z) \leq \overline{r}(z)$ *for all* $\xi \in U$ *and*

$$\overline{r}(z) - \underline{r}(z) + diam(U) \leq \frac{1}{2}\left(\sqrt{n+3} - \sqrt{n-1}\right)$$

for all $z \leq \min\{\hat{f}(x) : \ x \in \text{Round}(x^*_{CRC})\}$.
This implies that $\hat{f}$ *has the rounding property.*

Generalization

Corollary

Let

$$\mathcal{L}^{\xi}_{\leq}(z) = B_{x^*(\xi)}(r^{\xi}(z)) := \{x \in \mathbb{R}^n : \ \|x - x^*(\xi)\|_2 \leq r^{\xi}(z)\}$$

with $\underline{r}(z) \leq r^{\xi}(z) \leq \overline{r}(z)$ *for all* $\xi \in U$,
$u := diam(\{x^*(\xi) : \ \xi \in U\})$ *and*

$$\overline{r}(z) - \underline{r}(z) + u \leq \frac{1}{2}\left(\sqrt{n+3} - \sqrt{n-1}\right)$$

for all $z \leq \min\{\hat{f}(x) : \ x \in \text{Round}(x^*_{CRC})\}$.
This implies that $\hat{f}$ *has the rounding property.*

(Bary)

$$\xi = (\widetilde{A}_1, \dots, \widetilde{A}_5) \in \bigtimes_{i=1}^{5} \left([A_{i1} - \epsilon, A_{i1} + \epsilon] \times [A_{i2} - \epsilon, A_{i2} + \epsilon]\right) =: U$$

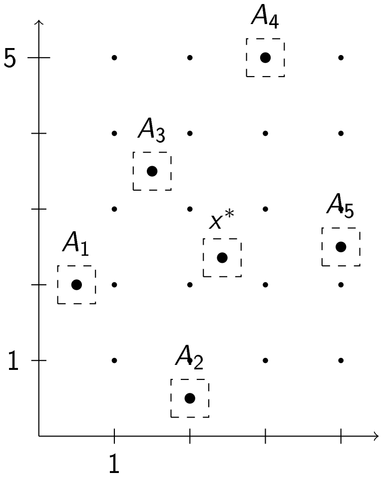
$$\Rightarrow \quad x^*(\xi) = \frac{1}{\Omega} \left(\sum_{i=1}^{5} \omega_i \widetilde{A}_{i1}, \sum_{i=1}^{5} \omega_i \widetilde{A}_{i2} \right)$$

where $\Omega = \sum_{i=1}^{5} \omega_i$.

$$\Rightarrow \qquad x^*(\xi) \in [x_1^* - \epsilon, x_1^* + \epsilon] \times [x_1^* - \epsilon, x_2^* + \epsilon]$$

where x^* is the continuous minimum of the nominal problem.
$\Rightarrow u := diam(\{x^*(\xi) : \ \xi \in U\}) = \sqrt{2}\epsilon.$

(Bary)



(Bary)

Eyster, White '73:

$$r^{\xi}(z) = \left[\frac{z}{W} + (x_1^*(\xi))^2 + (x_2^*(\xi))^2 - \frac{1}{W} \sum_{m=1}^{5} \omega_m (\widetilde{A}_{m1}^2 + \widetilde{A}_{m2}^2) \right]^{1/2}$$

$$\Rightarrow \quad \overline{r}(z) = \left[\frac{z}{W} + (x_1^* + \epsilon)^2 + (x_2^* + \epsilon)^2 - \frac{1}{W} \sum_{m=1}^{5} \omega_m ((A_{m1} - \epsilon)^2 + (A_{m2} - \epsilon)^2) \right]^{1/2}$$

$$\Rightarrow \quad \underline{r}(z) = \left[\frac{z}{W} + (x_1^* - \epsilon)^2 + (x_2^* - \epsilon)^2 - \frac{1}{W} \sum_{m=1}^{5} \omega_m ((A_{m1} + \epsilon)^2 + (A_{m2} + \epsilon)^2) \right]^{1/2}$$

$$\begin{aligned} \min\{\hat{f}(x) : \ x \in \text{Round}(x^*(\xi)\} &= \min\{\hat{f}(x) : \ x \in \text{Round}(x^*)\} \qquad \text{for } \epsilon < \frac{5}{14} \\ &= 13\epsilon^2 + 30\epsilon + 26.5. \end{aligned}$$

$$\Rightarrow \ \overline{r}(13\epsilon^2 + 30\epsilon + 26.5) - \underline{r}(13\epsilon^2 + 30\epsilon + 26.5) + \sqrt{2}\epsilon \leq \frac{1}{2}(\sqrt{5} - 1) \quad \text{for } \epsilon < 0.00894$$

Outline

Constraints

Problem:

$$
\begin{array}{lll}
(IP(\xi)) & \min & f(x,\xi) \\
 & \text{s.t.} & x \in X(\xi) \\
 & & x \in \mathbb{Z}^n
\end{array}
$$

$$\downarrow$$

Constraints

Problem:

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\begin{array}{lll}
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 & & x \in \mathbb{Z}^n
\end{array}
$$

$$\downarrow$$

Robust Counterpart:

$$
\begin{array}{lll}
(IRC) & \min & \hat{f}(x) \\
 & \text{s.t.} & x \in \bigcap_{\xi \in U} X(\xi) \\
 & & x \in \mathbb{Z}^n
\end{array}
$$

Constraints

Problem:

$$(IP(\xi)) \quad \min \quad f(x,\xi) \quad \text{s.t.} \quad x \in X(\xi), \quad x \in \mathbb{Z}^n$$

$$\downarrow$$

Robust Counterpart:

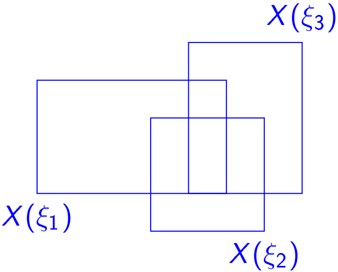
$$(IRC) \quad \min \quad \hat{f}(x) \quad \text{s.t.} \quad x \in \bigcap_{\xi \in U} X(\xi), \quad x \in \mathbb{Z}^n$$

$$\longrightarrow$$

Relaxation:

$$(CRC) \quad \min \quad \hat{f}(x) \quad \text{s.t.} \quad x \in \bigcap_{\xi \in U} X(\xi)$$

Constraints



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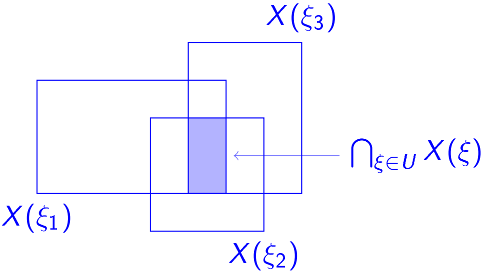

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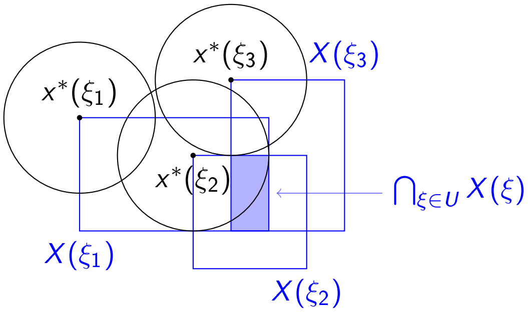

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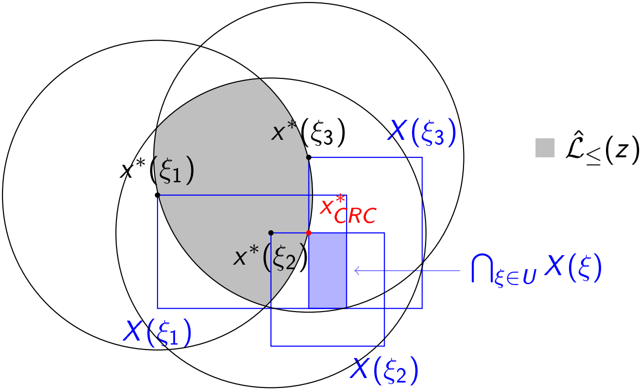
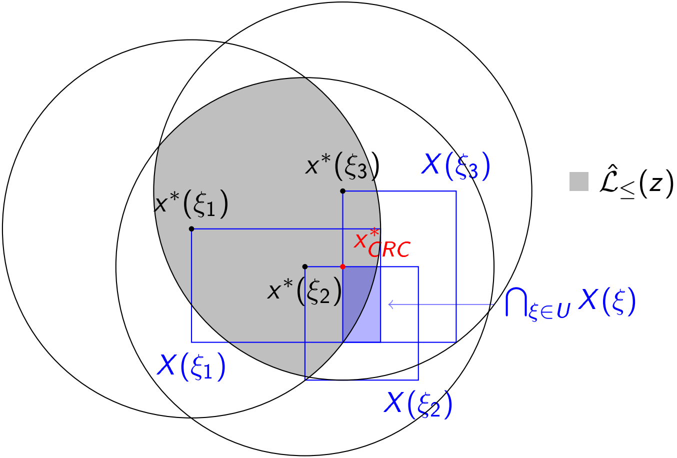


Constraints



Adjustable Robustness: Ben-Tal, Nemirovski '04

Uncertain Problem:

$$(IP(\xi)) \quad \min \quad f(x,\xi) \quad \text{s.t.} \quad x \in \mathbb{Z}^n$$

Adjustable Robustness: Ben-Tal, Nemirovski '04

Now:

$$
\begin{array}{lll}
(IP(\xi)) & \min & f(u,v,\xi) \\
 & \text{s.t.} & (u,v) \in \mathbb{Z}^{n_u} \times \mathbb{Z}^{n_v}
\end{array}
$$

Adjustable Robustness: Ben-Tal, Nemirovski '04

Now:

$$
\begin{array}{lll}
(IP(\xi)) & \min & f(u,v,\xi) \\
 & \text{s.t.} & (u,v) \in \mathbb{Z}^{n_u} \times \mathbb{Z}^{n_v}
\end{array}
$$

u = here-and-now variables
v = wait-and-see variables

Adjustable Robustness: Ben-Tal, Nemirovski '04

Now:

$$
\begin{array}{llll}
(IP(\xi)) & \min & f(u,v,\xi) \\
 & \text{s.t.} & (u,v) \in \mathbb{Z}^{n_u} \times \mathbb{Z}^{n_v}
\end{array}
$$

$$
\begin{array}{llll}
(jRC) & \min & \sup_{\xi \in U} \left(\inf_{v \in \mathbb{Z}^{n_v}} f(u,v,\xi) \right) & =: \; \sup_{\xi \in U} z(u,\xi) \\
 & \text{s.t.} & u \in \mathbb{Z}^{n_u}
\end{array}
$$

where $z(u,\xi) = \inf_{v \in \mathbb{Z}^{n_v}} f(u,v,\xi)$.

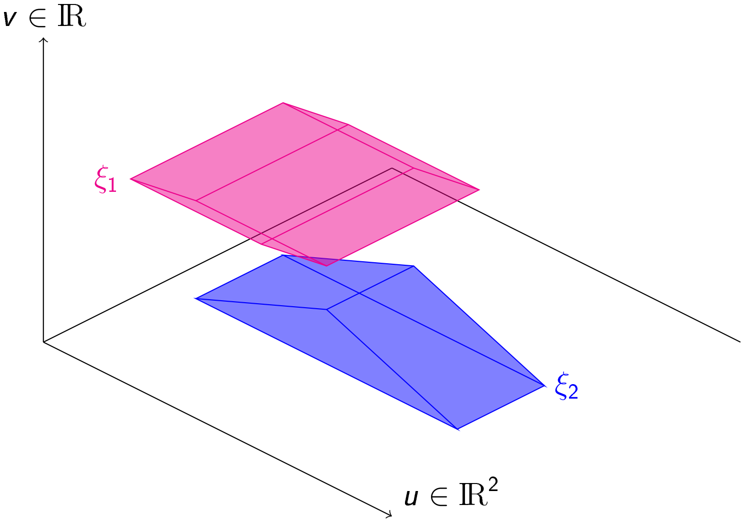
Level sets

$$
\begin{array}{lll}
(jRC) & \min & \hat{z}(u) \\
 & \text{s.t.} & u \in \mathbb{Z}^{n_u}
\end{array}
$$

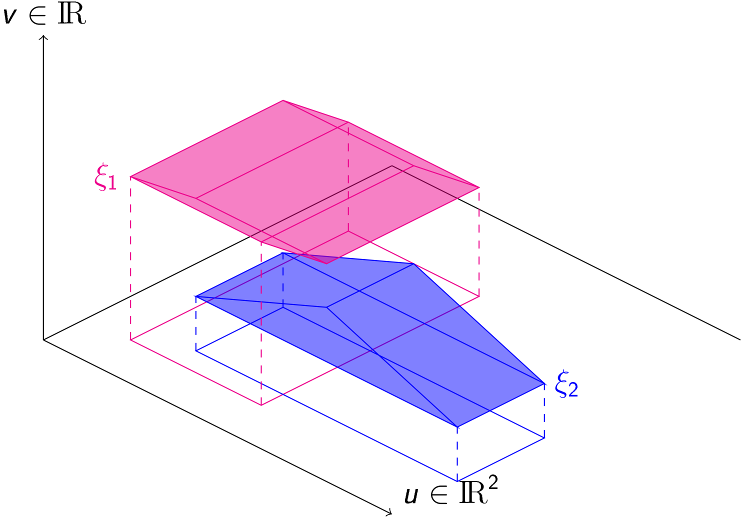
where $\hat{z}(u) := \sup_{\xi \in U} z(u,\xi)$ and $z(u,\xi) = \inf_{v \in \mathbb{Z}^{n_v}} f(u,v,\xi)$.

$$
\begin{aligned}
\mathcal{L}_{\leq,f}(w) &= \{(u,v,\xi):\ f(u,v,\xi) \leq w\} \\
\mathcal{L}_{\leq,z}(w) &= \{(u,\xi):\ \exists v:\ f(u,v,\xi) \leq w\} \\
&= Proj_{(\mathbb{R}^{n_u} \times \{0\})}(\mathcal{L}_{\leq,f}(w)) \\
\hat{\mathcal{L}}_{\leq}(w) &= \bigcap_{\xi \in U} \mathcal{L}_{\leq,z}(w) \\
&= \bigcap_{\xi \in U} Proj_{(\mathbb{R}^{n_u} \times \{0\})}(\mathcal{L}_{\leq,f}(w)).
\end{aligned}
$$

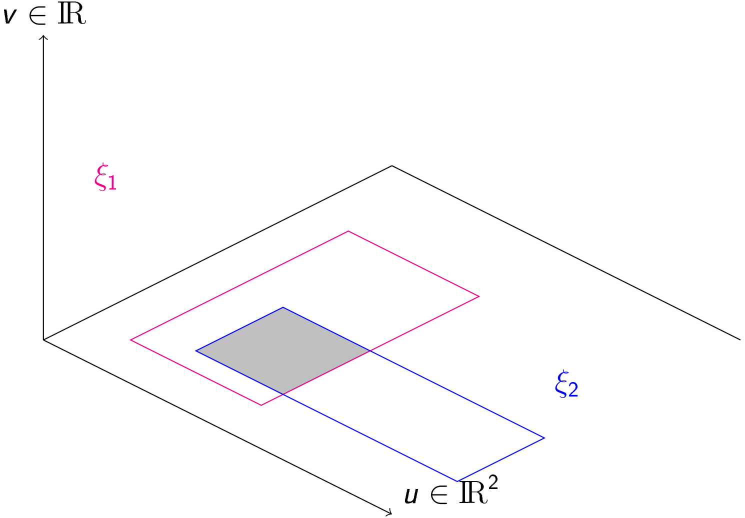
Example



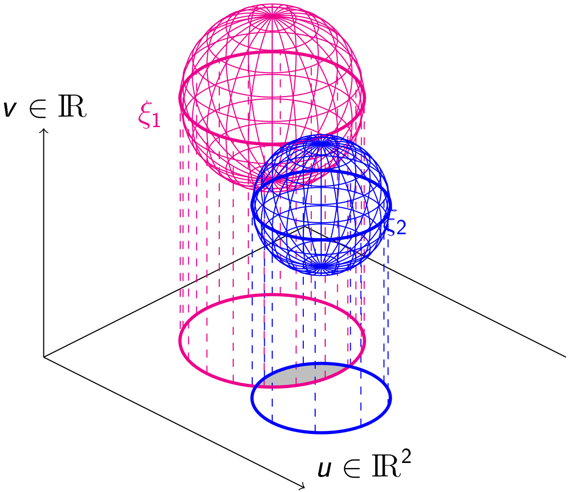
Example



Example



Example



Outline

Conclusion

Original Problem:

$$(IP(\xi)) \quad \min \quad f(x,\xi) \qquad \text{s.t.} \quad x \in \mathbb{Z}^n$$

$$\downarrow$$

Robust Counterpart:

$$(IRC) \quad \min \quad \hat{f}(x) \qquad \text{s.t.} \quad x \in \mathbb{Z}^n$$

$$\longrightarrow$$

Relaxation:

$$(CRC) \quad \min \quad \hat{f}(x) \qquad \text{s.t.} \quad x \in \mathbb{R}^n$$

$$\longleftarrow$$

Rounding Property

where $\hat{f}(x) := \sup_{\xi \in U} f(x,\xi)$.

Open Questions

- Find a better bound for ϵ in (Bary).
- Are there other uncertainty sets and/or level sets for which we can prove the rounding property?
- What do the level sets look like for other kind of robustness concepts?
- ...

Thank you for your attention.

- A. Ben-Tal, A. Goryashko, E. Guslitzer and A. Nemirovski, *Adjustable robust solutions of uncertain linear programs*, Mathematical Programming, 99, 2, 2004.
- A. Ben-Tal and A. Nemirovski, *Robust Convex Optimization*, Mathematics of Operations Research, Vol. 23, No. 4, 1998.
- A. Ben-Tal and A. Nemirovski, *Robust solutions of uncertain linear programs*, Operations Research Letters, 25, 1999.
- J.W. Eyster and J.A. White, *Some properties of the squared Euclidean distance location problem*, AIIE Transaction 5, 275-280, 1973.
- R. Hübner, A. Schöbel, *When is rounding allowed? A new approach to integer nonlinear optimization*, submitted, 2011
- L. V. Snyder, *Facility Location Under Uncertainty: A Review*, IIE Transactions 38(7), 2006.
- A.L. Soyster, *Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming*, Operations Research, Vo. 21, No. 5, 1973.