Robustness in Multiobjective Optimization

Jonas Ide

University of Göttingen

joint work with
Matthias Ehrgott and Anita Schöbel

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Definition (Multiobjective optimization problem)

Given a feasible set $\mathbb{X} \subset \mathbb{R}^n$ *and a function* $f : \mathbb{R}^n \to \mathbb{R}^k$, *a multiobjective optimization problem is given by*

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in \mathbb{X} \end{array}$$

In Multiobjective Optimization one searches for the set of *nondominated* points $f(\overline{x})$ with $\overline{x} \in \mathbb{X}$.
The according solution $\overline{x}$ is called *efficient*.

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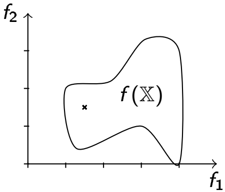

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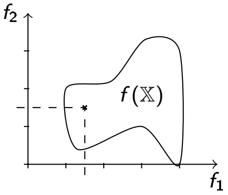

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Definition (Uncertain optimization problem)

Given an uncertainty set $\mathcal{U}$, a feasible set $\mathbb{X} \subset \mathbb{R}^n$ and a function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, an uncertain optimization problem $\mathcal{P}(\mathcal{U})$ is given by the family of all problems

$$\mathcal{P}(\xi) \quad \begin{array}{ll} \min & f(x,\xi) \\ s.t. & x \in \mathbb{X} \end{array}$$

with $\xi \in \mathcal{U}$.

The question arises

When do we call a solution $\overline{x} \in \mathbb{X}$ *robust* optimal?

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Different concepts of robustness (e.g. Bertsimas and Sim, 2004, Ben-Tal et al., 2009, etc.)

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  $\min_{x \in \mathbb{X}} \sup_{\xi \in \mathcal{U}} f(x, \xi)$

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- many other concepts

Definition (Uncertain multiobjective problem)

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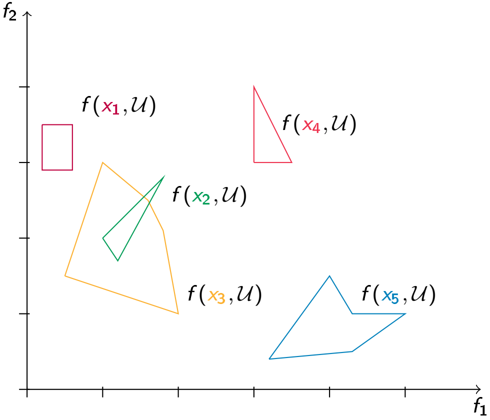
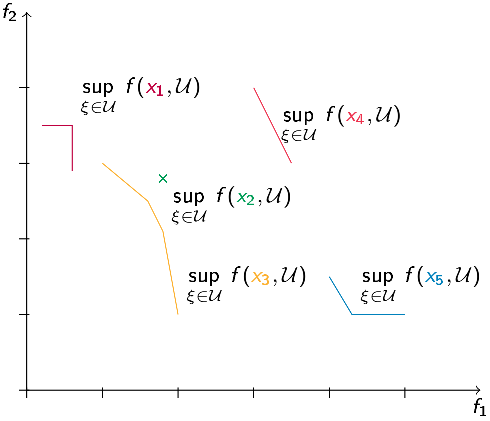


Which of these solutions do we call robust efficient?



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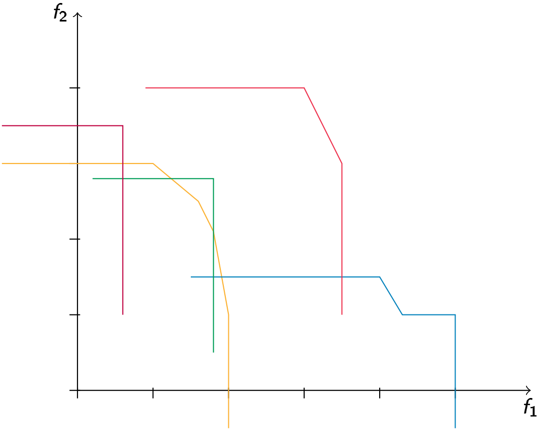

We will call those $x \in \mathbb{X}$ robust efficient, where $f(x, \mathcal{U})$ is *nondominated*.

Definition (Robust efficiency)

Given an uncertain multiobjective problem $\mathcal{P}(\mathcal{U})$ *we call a solution* $\overline{x} \in \mathbb{X}$ robust efficient,

if there is no $x' \in \mathbb{X} \setminus \{\overline{x}\}$ *such that*

$$f(x', \mathcal{U}) \subseteq f(\overline{x}, \mathcal{U}) - \mathbb{R}^k_{\geq}$$

Definition (Robust efficiency)

Given an uncertain multiobjective problem $\mathcal{P}(\mathcal{U})$ *we call a solution* $\overline{x} \in \mathbb{X}$ robust strictly efficient,

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Definition (Robust efficiency)

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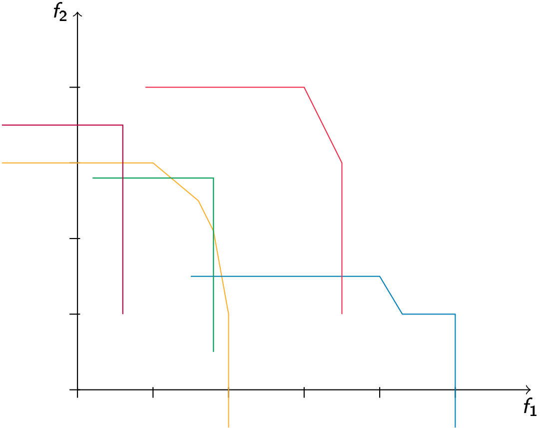

The orange, blue, green and purple solutions are robust strictly efficient, the red one is not even robust weakly efficient.

Properties

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  - First idea: Find solutions by solving a robust single-objective problem

Properties

- For $|\mathcal{U}| = 1$ these definitions reduce to the definition of efficiency
- For $k = 1$ the definition of robust weakly efficiency reduces to the definition of strictly robust optimality

Question

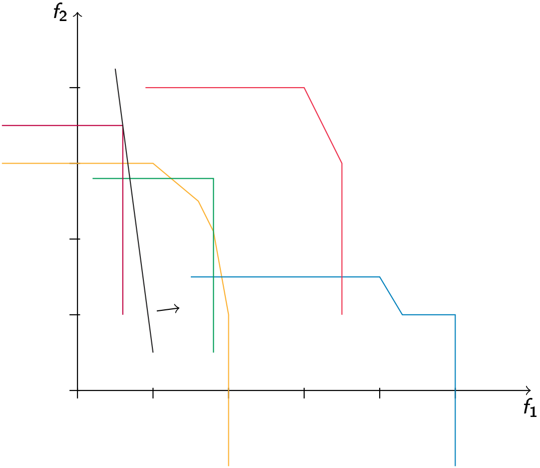
- How to calculate robust efficient solutions?
  - First idea: Find solutions by solving a robust single-objective problem
  - Second idea: Find solutions by solving a deterministic multi-objective problem

Theorem

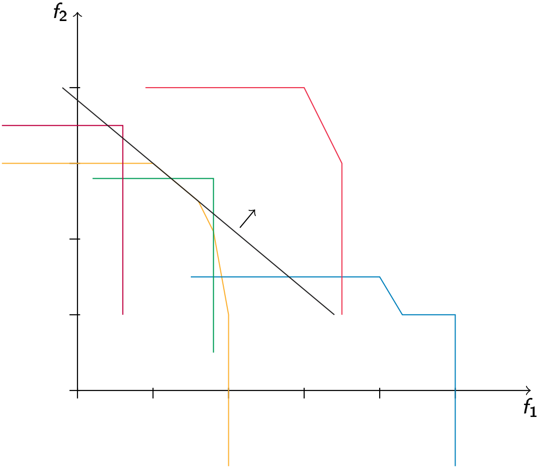
If $\overline{x} \in \mathbb{X}$ is the unique minimizer of

$$\sup_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x, \xi)$$

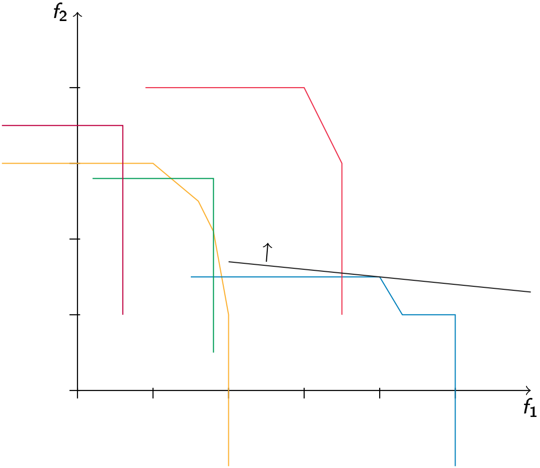
over $\mathbb{X}$ for some $\lambda \in \mathbb{R}^k_{\geq}$, $\overline{x}$ is robust strictly efficient.



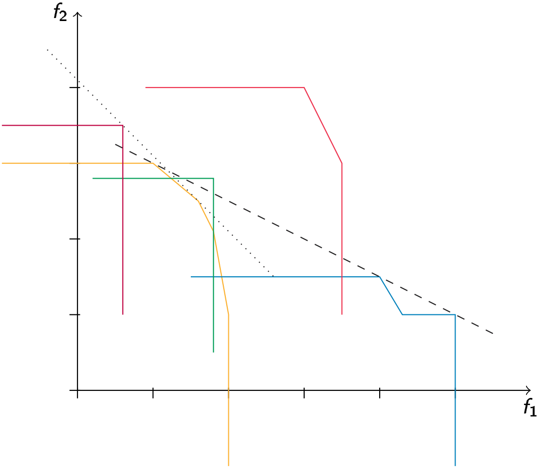
The purple solution is robust strictly efficient.



The orange solution is robust strictly efficient.



The blue solution is robust strictly efficient.



The green robust strictly efficient solution is no optimal solution for any scalarization problem.

Theorem

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over $\mathbb{X}$ for some $\lambda \in \mathbb{R}^k_{\geq}$, $\overline{x}$ is robust strictly efficient.

Theorem

If $\max\limits_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x, \xi)$ exists for all $x \in \mathbb{X}$ and $\overline{x} \in \mathbb{X}$ is a minimizer of

$$\max_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x, \xi)$$

over $\mathbb{X}$ for some $\lambda \in \mathbb{R}^k_{\geq}$, then $\overline{x}$ is robust weakly efficient.

Definition

$$
\begin{array}{lll}
(\epsilon\mathcal{RC}) & \min & \sup\limits_{\xi\in\mathcal{U}} f_i(x,\xi) \\
& \text{s.t.} & f_j(x,\xi) \leq \epsilon_j \ \forall\, j \neq i,\ \forall\, \xi \in \mathcal{U} \\
& & x \in \mathbb{X}
\end{array}
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Theorem

Given a problem $\mathcal{P}(\mathcal{U})$.

a) *If $\overline{x} \in \mathbb{X}$ is the unique optimal solution to $\epsilon\mathcal{RC}$ for some i, then it is robust strictly efficient.*

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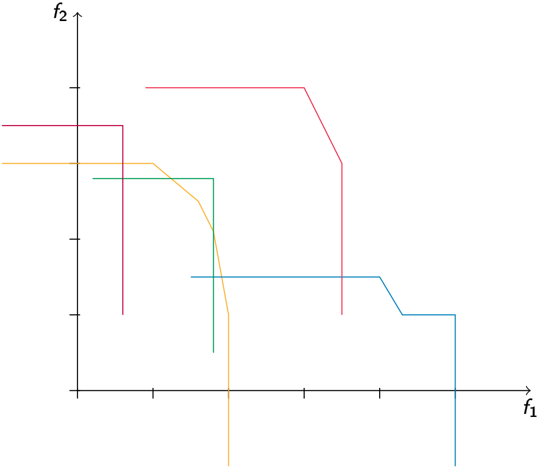
Theorem

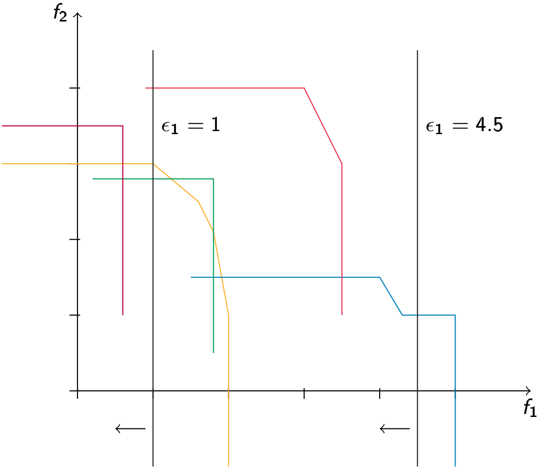
Given a problem $\mathcal{P}(\mathcal{U})$.

a) *If $\overline{x} \in \mathbb{X}$ is the unique optimal solution to $\epsilon\mathcal{RC}$ for some i, then it is robust strictly efficient.*

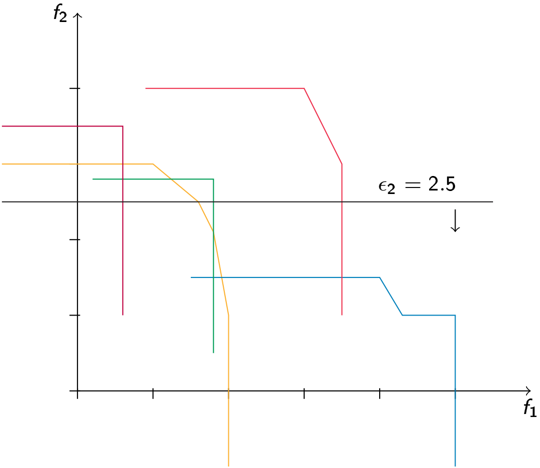
b) *If $\overline{x} \in \mathbb{X}$ is an optimal solution to $\epsilon\mathcal{RC}$ for some i and $\max\limits_{\xi\in\mathcal{U}} f_i(x,\xi)$ exists for all $x \in \mathbb{X}$, then $\overline{x}$ is robust weakly efficient.*



f_2
f_1



The green solution minimizes $f_2(x, \xi)$ over $\{x \in \mathbb{X} : \; f_1(x, \xi) \leq 4.5 \; \forall \xi \in \mathcal{U}\}$,
the purple solution minimizes $f_2(x, \xi)$ over $\{x \in \mathbb{X} : \; f_1(x, \xi) \leq 1 \; \forall \xi \in \mathcal{U}\}$,



The blue solution minimizes $f_1(x,\xi)$ over $\{x \in \mathbb{X} : \; f_2(x,\xi) \leq 2.5 \; \forall \xi \in \mathcal{U}\}$

The orange solution cannot be found with the ϵ-constraint method.

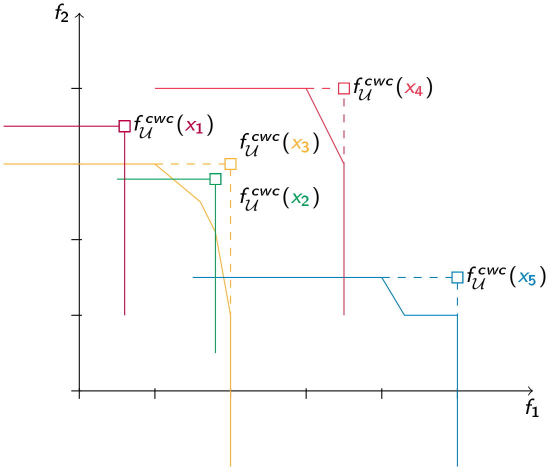
Definition

We formulate a new problem

$$(\mathcal{CWC})\ \min_{x \in \mathbb{X}}\ f_{\mathcal{U}}^{cwc}(x)$$

where

$$f_{\mathcal{U}}^{cwc}(x) := \begin{pmatrix} \sup\limits_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \sup\limits_{\xi \in \mathcal{U}} f_2(x, \xi) \\ \vdots \\ \sup\limits_{\xi \in \mathcal{U}} f_k(x, \xi) \end{pmatrix}$$



f_2
f_1
$f_{\mathcal{U}}^{cwc}(x_1)$
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$f_{\mathcal{U}}^{cwc}(x_3)$
$f_{\mathcal{U}}^{cwc}(x_4)$
$f_{\mathcal{U}}^{cwc}(x_5)$

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(1) *If $\overline{x} \in \mathbb{X}$ is a strictly efficient solution for $(\mathcal{CWC})$, then it is robust strictly efficient for $\mathcal{P}(\mathcal{U})$.*

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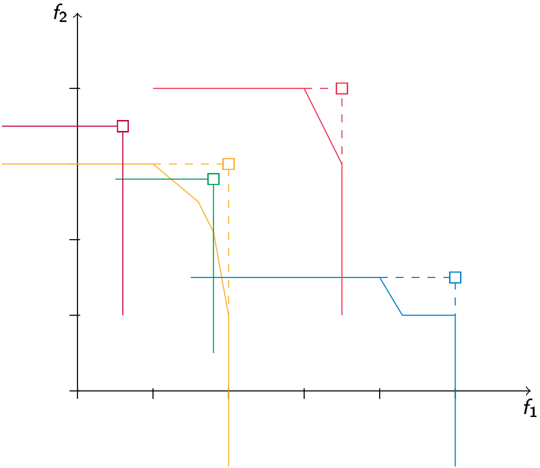
(2) *If $\max\limits_{\xi \in \mathcal{U}} f_i(x,\xi)$ exists for all $i = 1, \ldots, k$ and $x \in \mathbb{X}$ and $\overline{x}$ is weakly efficient for $(\mathcal{CWC})$, it is robust weakly efficient for $\mathcal{P}(\mathcal{U})$.*



The strictly efficient solutions of $(\mathcal{CWC})$ are the purple, green and blue solutions. The orange solution cannot be found this way.

ϵ-constraint method and $(\mathcal{CWC})$-method yield exactly the same solutions.

Applications

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- RTG 1703 "Resource Efficiency in Corporate Networks"

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Theoretical questions

- Some open theoretical questions (i.e. convexity for weighted sum scalarization)
- Other concepts of robustness / different interpretation of the supremum

Thank you for your attention!