An LP-Based Heuristic for Flexible Job Shop Scheduling

Morten Tiedemann

Institute for Numerical and Applied Mathematics
University of Goettingen

Summer School on Optimization

August 27 2012

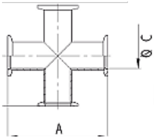
PFEIFFER VACUUM

Manufacturer of vacuum chambers, vacuum systems and special components for the vacuum industry

PFEIFFER VACUUM

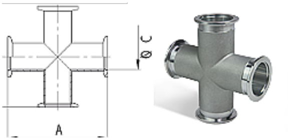
Manufacturer of vacuum chambers, vacuum systems and special components for the vacuum industry

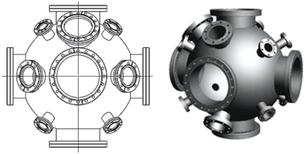
Basic Data



job	operation	processing time	valid machines
ISO-KF 4-way cross	welding (simple)	180	|10||53||55||62|
ISO-KF 4-way cross	cleaning	30	|6||8||9||54||58|
ISO-KF 4-way cross	sealing	120	|1||4||15|
ISO-KF 4-way cross	leak detection	60	|43|
ISO-KF 4-way cross	labeling	30	|71||72|
ISO-KF 4-way cross	shipping	60	|66||69||70|

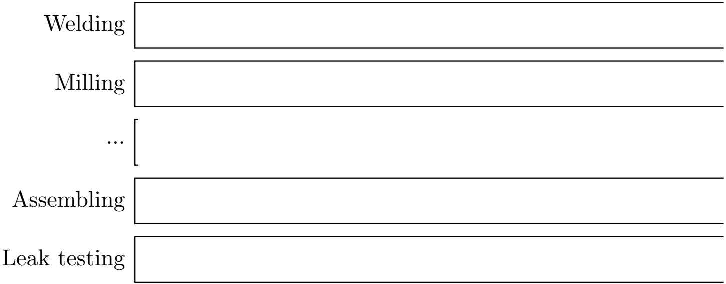
Basic Data



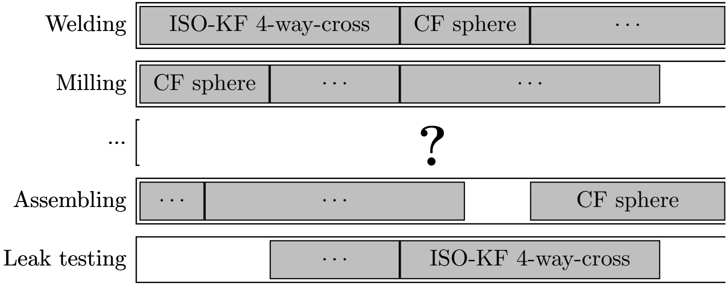


<table>
<tr><th>job</th><th>operation</th><th>processing time</th><th>valid machines</th></tr>
<tr><td>ISO-KF 4-way cross</td><td>welding (simple)</td><td>180</td><td>|10||53||55||62|</td></tr>
<tr><td>ISO-KF 4-way cross</td><td>cleaning</td><td>30</td><td>|6||8||9||54||58|</td></tr>
<tr><td>ISO-KF 4-way cross</td><td>sealing</td><td>120</td><td>|1||4||15|</td></tr>
<tr><td>ISO-KF 4-way cross</td><td>leak detection</td><td>60</td><td>|43|</td></tr>
<tr><td>ISO-KF 4-way cross</td><td>labeling</td><td>30</td><td>|71||72|</td></tr>
<tr><td>ISO-KF 4-way cross</td><td>shipping</td><td>60</td><td>|66||69||70|</td></tr>
<tr><td></td><td>...</td><td>...</td><td></td></tr>
<tr><td>CF sphere</td><td>welding (plasma)</td><td>480</td><td>|53||55|</td></tr>
<tr><td>CF sphere</td><td>welding (simple)</td><td>300</td><td>|10||53||55||62|</td></tr>
<tr><td></td><td>...</td><td>...</td><td></td></tr>
</table>

Statement of the Problem



Statement of the Problem



What is an optimal assignment of operations to machines and ordering of operations on the machines?

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$
- n jobs $J_1, \ldots, J_n$

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$
- n jobs $J_1, \ldots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \ldots, n_i$

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$
- n jobs $J_1, \ldots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \ldots, n_i$
- Fixed sequence $O_{i,1} \to O_{i,2} \to \cdots \to O_{i,n_i}$

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \dots, M_m\}$
- n jobs $J_1, \dots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \dots, n_i$
- Fixed sequence $O_{i,1} \rightarrow O_{i,2} \rightarrow \cdots \rightarrow O_{i,n_i}$
- Operation $O_{i,j}$ has to be processed for $p_{i,j,k}$ time units on a machine from its set of valid machines

$$M_{i,j} \subseteq \{M_1, \dots, M_m\}$$

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$
- n jobs $J_1, \ldots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \ldots, n_i$
- Fixed sequence $O_{i,1} \to O_{i,2} \to \cdots \to O_{i,n_i}$
- Operation $O_{i,j}$ has to be processed for $p_{i,j,k}$ time units on a machine from its set of valid machines

$$M_{i,j} \subseteq \{M_1, \ldots, M_m\}$$

- Operations have to be processed preemptively

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$
- n jobs $J_1, \ldots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \ldots, n_i$
- Fixed sequence $O_{i,1} \rightarrow O_{i,2} \rightarrow \cdots \rightarrow O_{i,n_i}$
- Operation $O_{i,j}$ has to be processed for $p_{i,j,k}$ time units on a machine from its set of valid machines

$$M_{i,j} \subseteq \{M_1, \ldots, M_m\}$$

- Operations have to be processed preemptively
- Each machine processes at most one operation at a time

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \ldots, M_m\}$
- n jobs $J_1, \ldots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \ldots, n_i$
- Fixed sequence $O_{i,1} \to O_{i,2} \to \cdots \to O_{i,n_i}$
- Operation $O_{i,j}$ has to be processed for $p_{i,j,k}$ time units on a machine from its set of valid machines

$$M_{i,j} \subseteq \{M_1, \ldots, M_m\}$$

- Operations have to be processed preemptively
- Each machine processes at most one operation at a time
- Objective $C_{\max} := \max_{i=1,\ldots,n} C_i$

Flexible Job Shop Scheduling

- Set of machines $M = \{M_1, \dots, M_m\}$
- n jobs $J_1, \dots, J_n$
- Job J_i consists of n_i operations $O_{i,j}$, $j = 1, \dots, n_i$
- Fixed sequence $O_{i,1} \to O_{i,2} \to \cdots \to O_{i,n_i}$
- Operation $O_{i,j}$ has to be processed for $p_{i,j,k}$ time units on a machine from its set of valid machines

$$M_{i,j} \subseteq \{M_1, \dots, M_m\}$$

- Operations have to be processed preemptively
- Each machine processes at most one operation at a time
- Objective $C_{\max} := \max_{i=1,\dots,n} C_i$

Feasible Schedule

A schedule $S = (S_{i,j})$ is defined by the starting time of all operations and a machine assignment $(m_{i,j})$.

Feasible Schedule

A schedule $S = (S_{i,j})$ is defined by the starting time of all operations and a machine assignment $(m_{i,j})$. A *feasible* schedule has to respect the following properties:

$$S_{i,j} + p_{i,j,m_{i,j}} \leq S_{i,j+1} \quad \text{for all jobs } i = 1, \ldots, n, \\ \text{operations } j = 1, \ldots, n_i - 1$$

Feasible Schedule

A schedule $S = (S_{i,j})$ is defined by the starting time of all operations and a machine assignment $(m_{i,j})$. A *feasible* schedule has to respect the following properties:

$$S_{i,j} + p_{i,j,m_{i,j}} \leq S_{i,j+1} \quad \text{for all jobs } i = 1, \ldots, n, \text{ operations } j = 1, \ldots, n_i - 1$$

$$S_{i,j} + p_{i,j,m_{i,j}} \leq S_{k,l} \vee S_{k,l} + p_{k,l,m_{k,l}} \leq S_{i,j} \quad \text{for all pairs } O_{i,j},\ O_{k,l} \text{ of operations with } m_{i,j} = m_{k,l}$$

Complexity and Approximation Algorithms

Modelling

LP-Based Heuristic

Complexity

The flexible job shop scheduling problem is $\mathcal{NP}$-complete.

Complexity

The flexible job shop scheduling problem is $\mathcal{NP}$-complete, even for ...

- ... 3 jobs and 3 machines.

Complexity

The flexible job shop scheduling problem is $\mathcal{NP}$-complete, even for ...

- ... 3 jobs and 3 machines.
- ... 2 jobs and arbitrarily many machines.

Complexity

The flexible job shop scheduling problem is $\mathcal{NP}$-complete, even for ...

- ... 3 jobs and 3 machines.
- ... 2 jobs and arbitrarily many machines.

The flexible job shop scheduling problem is solvable in polynomial time for two jobs and identical processing times.

Approximation Algorithms

Consider the problem

$$FJ \,|\, p_{i,k} = p_i \,|\, C_{\max}$$

Approximation Algorithms

Consider the problem

$$FJ \mid p_{i,k} = p_i \mid C_{\max}$$

Theorem (1)

There is an LP-based heuristic LPH for $FJ \mid p_{i,k} = p_i \mid C_{\max}$ *with*

$$val(LPH) \leq \mu \left(\frac{2m}{\min_j |M_j|} + 1 \right) \left(1 - \frac{m}{\max_j |M_j| \, n + m} \right) val(OPT).$$

Approximation Algorithms

Consider the problem

$$FJ \mid p_{i,k} = p_i \mid C_{\max}$$

Theorem (2)

There is an polynomial-time algorithm ALG for $FJ \mid p_{i,k} = p_i \mid C_{\max}$ *with*

$$val(ALG) \leq \rho \left(3 - \frac{1}{p_{\max}}\right) val(OPT),$$

where $\rho = \frac{\log m}{\log \log m} \log(\min\{m\mu, p_{\max}\})$.

Modeling

$$\begin{array}{lll} \min & C_{\max} & \\ \text{s.t.} & C_{\max} - \sum_{k \in M_i} x_{i,k} p_{i,k} \geq S_i & \text{for all } i \in O : P(i) = n_{J(i)} \end{array}$$

Modeling

$$\begin{aligned}
\min \quad & C_{\max} & \\
\text{s.t.} \quad & C_{\max} - \sum_{k \in M_i} x_{i,k} p_{i,k} \geq S_i & \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{k \in M_i} x_{i,k} = 1 & \text{for all } i \in O
\end{aligned}$$

Modeling

$$
\begin{aligned}
\min \quad & C_{\max} & \\
\text{s.t.} \quad & C_{\max} - \sum_{k \in M_i} x_{i,k} p_{i,k} \geq S_i & \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{k \in M_i} x_{i,k} = 1 & \text{for all } i \in O \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j & \text{for all } (i,j) \in C
\end{aligned}
$$

Modeling

$$
\begin{aligned}
\min \quad & C_{\max} \\
\text{s.t.} \quad & C_{\max} - \sum_{k \in M_i} x_{i,k} p_{i,k} \geq S_i && \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{k \in M_i} x_{i,k} = 1 && \text{for all } i \in O \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j && \text{for all } (i,j) \in C \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j \quad \vee \\
& S_j + \sum_{k \in M_j} x_{j,k} p_{j,k} \leq S_i && \text{for all } (i,j) \in D(x)
\end{aligned}
$$

Modeling

$$
\begin{aligned}
\min \quad & C_{\max} & \\
\text{s.t.} \quad & C_{\max} - \sum_{k \in M_i} x_{i,k} p_{i,k} \geq S_i & \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{k \in M_i} x_{i,k} = 1 & \text{for all } i \in O \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j & \text{for all } (i,j) \in C \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j \quad \vee & \\
& S_j + \sum_{k \in M_j} x_{j,k} p_{j,k} \leq S_i & \text{for all } (i,j) \in D(x) \\
& S_i \geq 0 & \text{for all } i \in O \\
& x_{i,k} \in \{0,1\} & \text{for all } k \in M_i;\ i \in O
\end{aligned}
$$

Modeling

$$
\begin{aligned}
\min \quad & C_{\max} && \\
\text{s.t.} \quad & C_{\max} - \sum_{k \in M_i} x_{i,k} p_{i,k} \geq S_i && \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{k \in M_i} x_{i,k} = 1 && \text{for all } i \in O \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j && \text{for all } (i,j) \in C \\
& S_i + \sum_{k \in M_i} x_{i,k} p_{i,k} \leq S_j \quad \vee && \\
& S_j + \sum_{k \in M_j} x_{j,k} p_{j,k} \leq S_i && \text{for all } (i,j) \in D(x) \\
& S_i \geq 0 && \text{for all } i \in O \\
& x_{i,k} \in \{0,1\} && \text{for all } k \in M_i;\ i \in O
\end{aligned}
$$

Additional Assumptions

Additional Assumptions

Processing Times

Identical processing times $p_{i,k}$ for an operation i on all valid machines.

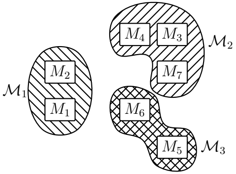
Additional Assumptions

Processing Times

Identical processing times $p_{i,k}$ for an operation i on all valid machines.

Machine Structure

Classification of machines $M_1, \ldots, M_m$ in disjunctive machine groups $\mathcal{M}_1, \ldots, \mathcal{M}_{\tilde{m}}$



STEF Structured Time-Expanded Formulation

$$
\begin{aligned}
\min \quad & C_{\max} && && \text{(STEF)} \\
\text{s.t.} \quad & \sum_{t=1}^{T} x_{i,t}(t+p_i) \leq C_{\max} && \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{t=1}^{T} x_{i,t}(t+p_i) \leq \sum_{t=1}^{T} x_{j,t}\, t && \text{for all } i,j \in C \\
& \sum_{t=1}^{T} x_{i,t} = 1 && \text{for all } i \in O \\
& x_{i,t} \in \{0,1\} && \text{for all } i \in O,\ t = 1,\dots,T.
\end{aligned}
$$

STEF Structured Time-Expanded Formulation

$$
\begin{aligned}
\min \quad & C_{\max} && && \text{(STEF)} \\
\text{s.t.} \quad & \sum_{t=1}^{T} x_{i,t}(t+p_i) \le C_{\max} && \text{for all } i \in O : P(i) = n_{J(i)} \\
& \sum_{t=1}^{T} x_{i,t}(t+p_i) \le \sum_{t=1}^{T} x_{j,t}\, t && \text{for all } i,j \in C \\
& \sum_{t=1}^{T} x_{i,t} = 1 && \text{for all } i \in O \\
& \sum_{i \in O} \sum_{\tau = t - p_i + 1}^{t} x_{i,\tau} \le |\mathcal{M}_j| && \text{for all } j = 1,\dots,\tilde{m};\ t = 1,\dots,T, \\
& x_{i,t} \in \{0,1\} && \text{for all } i \in O,\ t = 1,\dots,T.
\end{aligned}
$$

Algorithm 1 LP-Based Heuristic

1: Solve the linear relaxation of the model STEF. Let

$$x^* = \left(x^*_{i,t}\right)$$

be the solution of the linear relaxation.

Algorithm 2 LP-Based Heuristic

1: Solve the linear relaxation of the model STEF. Let

$$x^* = \left(x^*_{i,t}\right)$$

be the solution of the linear relaxation.

2: Define the starting times S_i for $i \in O$ by

$$S_i = \sum_{t=\alpha_i}^{\beta_i} x^*_{i,t} \cdot t$$

Algorithm 3 LP-Based Heuristic

1: Solve the linear relaxation of the model STEF. Let

$$x^* = \left(x^*_{i,t}\right)$$

be the solution of the linear relaxation.

2: Define the starting times S_i for $i \in O$ by

$$S_i = \sum_{t=\alpha_i}^{\beta_i} x^*_{i,t} \cdot t$$

3: Index the operations such that $S_1 \leq S_2 \leq \cdots \leq S_N$

Algorithm 4 LP-Based Heuristic

Solve the linear relaxation of the model STEF. Let

$$x^* = \left(x^*_{i,t}\right)$$

be the solution of the linear relaxation.

Define the starting times S_i for $i \in O$ by

$$S_i = \sum_{t=\alpha_i}^{\beta_i} x^*_{i,t} \cdot t$$

Index the operations such that $S_1 \leq S_2 \leq \cdots \leq S_N$

for all $i = 1, \ldots, N$ **do**

    Schedule operation i as early as possible

end for

LP-Based Heuristic

The choice of the starting times

$$S_i = \sum_{t=\alpha_i}^{\beta_i} x^*_{i,t} \cdot t$$

leads to

- $S_i + p_i - S_j = \sum_{t=\alpha_i}^{\beta_i} x_{i,t} \cdot t + p_i - \sum_{t=\alpha_j}^{\beta_j} x_{j,t} \cdot t \leq 0 \quad \text{for all } (i,j) \in C,$

LP-Based Heuristic

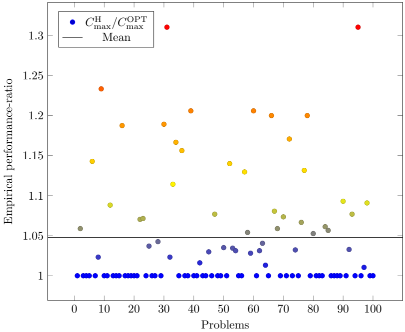
The choice of the starting times

$$S_i = \sum_{t=\alpha_i}^{\beta_i} x^*_{i,t} \cdot t$$

leads to

- $S_i + p_i - S_j = \sum_{t=\alpha_i}^{\beta_i} x_{i,t} \cdot t + p_i - \sum_{t=\alpha_j}^{\beta_j} x_{j,t} \cdot t \le 0 \quad \text{for all } (i,j) \in C,$
- $S_i + p_i = \sum_{t=\alpha_i}^{\beta_i} x_{i,t} t + p_i \le C_{\max} \quad \text{for all } i \in O : P(i) = n_{J(i)}.$

Results



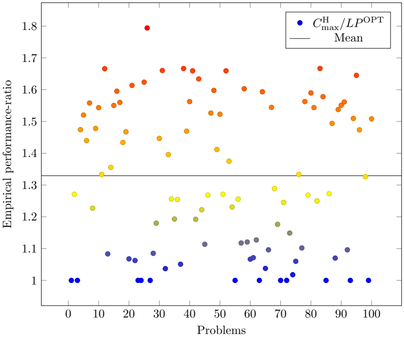


Figure: Empirical performance-ratio of the LP-basted heuristic

Results

instances	n	m	$\tilde{m}$	N	f	objective	ratio OPT	ratio LR	∅ CPU
small	6	4	2	20	0.8	$C_{\max}$	1.071	1.291	0.05
small	6	4	2	20	0.8	$\sum C_i$	1.061	1.092	0.06
large	60	30	5	200	0.8	$C_{\max}$	-	1.319	2.58
large	60	30	5	200	0.8	$\sum C_i$	-	1.064	4.20

Table: Computational Results for LP-based Heuristic

- Utilization of the information comprised in the IP-formulation

- Utilization of the information comprised in the IP-formulation
- LP-based heuristic produces close to optimal schedules

- Utilization of the information comprised in the IP-formulation
- LP-based heuristic produces close to optimal schedules