Summer School OptALI, Auckland
Lecture on Robust Optimization
Handout

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Let us consider a general optimization problem

$$(P) \qquad \min\{f(x) : F(x) \le 0, x \in \mathbb{R}^n\}$$

with functions $f : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^m$. If some parameters are unknown, the uncertain problem is written as

$$\mathrm{P}(\xi) \qquad \min\{f(x,\xi) : F(x,\xi) \le 0, x \in \mathbb{R}^n\}, \qquad \xi \in \mathcal{U}$$

indicating that both the objective function and the constraints may depend on some (unknown) parameters $\xi \in \mathcal{U} \subseteq \mathbb{R}^M$, or for short

$$P(\xi), \xi \in \mathcal{U}. \tag{1}$$

Often it is assumed that there is some *nominal (undisturbed) scenario* $\hat{\xi} \in \mathcal{U}$. Then $\mathrm{P}(\hat{\xi})$ is called the *nominal problem*.

$F(\xi) \subseteq \mathbb{R}^n$ denotes the set of feasible solutions for $P(\xi)$, i.e.

$$F(\xi) = \{x \in \mathbb{R}^n : F(x,\xi) \le 0\}.$$

The objective value of $P(\xi)$ (if it exists) is denoted by $f^*(\xi)$.

Strict robustness

A solution $x \in \mathbb{R}^n$ to the uncertain problem (1) is called *strictly robust* if it is feasible for any of the scenarios in $\mathcal{U}$, i.e. if

$$F(x,\xi) \le 0 \text{ for all } \xi \in \mathcal{U}.$$

Let $\mathrm{SR}(\mathcal{U}) \subseteq \mathbb{R}^n$ denote the set of strictly feasible solutions w.r.t $\mathcal{U}$.

a) minimizing the worst-case

The classical *strictly robust counterpart* looks at the worst case objective among all scenarios and is given as

$$\begin{array}{ll} (\mathrm{RC}) & \min \ \sup_{\xi \in \mathcal{U}} f(x,\xi) \\ & \text{s.t.} F(x,\xi) \le 0 \text{ for all } \xi \in \mathcal{U} \\ & x \in \mathbb{R}^n \end{array}$$

b) minimizing the worst-case regret

Here the worst-case deviation to the best possible solution (among all scenarios) is considered.

$$
\text{(RegC)} \qquad \begin{array}{c} \min \sup_{\xi \in \mathcal{U}} f(x,\xi) - f^*(\xi) \\ \text{s.t.} F(x,\xi) \leq 0 \text{ for all } \xi \in \mathcal{U} \\ x \in \mathbb{R}^n \end{array}
$$

Reliability

Let $\gamma \in \mathbb{R}^m_{\geq 0}$. A solution $x \in \mathbb{R}^n$ has **reliability** γ if

$$F(x,\xi) \leq \gamma \text{ for all } \xi \in \mathcal{U}.$$

Adjustable Robustness

Split the variables $x \in \mathbb{R}^n$ in

$$x = (u,v) \quad u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}$$

with $n_u + n_v = n$. The variables u have to be fixed before the scenario ξ is realized ("here and now" variables). The variables v can be adjusted after the real scenario ξ becomes known ("wait and see" variables). Then $u \in \mathbb{R}^{n_u}$ is **adjustable robust** for $(P(\xi), \xi \in \mathcal{U}$, if for any $\xi \in \mathcal{U}$ there exists some $v \in \mathbb{R}^{n_v}$ such that

$$F(u,v,\xi) \leq 0.$$

Light Robustness

Let $z^* = f^*(\hat{\xi})$ be the objective value of the nominal problem, and let $\rho \in \mathbb{R}_+$ be a given. Let

$$\text{NQ} = \{x \in \mathbb{R}^n : f(x,\hat{\xi}) \leq z^* + \rho\}$$

be the solutions with a given required standard (w.r.t nominal quality). We look for a solution $x \in \text{NQ}$ with minimal reliability $\|\gamma\|_1$.

Recovery Robustness

Define a class $\mathcal{A}$ of possible recovery algorithms where each algorithm $\mathsf{Alg} \in \mathcal{A}$ takes $x \in \mathbb{R}^n$ and a scenario $\xi \in \mathcal{U}$ as input and calculates a new solution $y \in \mathbb{R}^n$, i.e.,

$$y = \mathsf{Alg}(x,\xi).$$

A solution $x \in \mathbb{R}^n$ is **recovery robust** if there exists $\mathsf{Alg} \in \mathcal{A}$ such that

$$\mathsf{Alg}(x,\xi) \in F(\xi) \text{ for all } \xi \in \mathcal{U}.$$

The pair (x, Alg) is then denoted as **precaution**. A possible goal is to minimize the nominal objective value $f(x,\hat{\xi})$ among all recovery-robust solutions.